

# dNLS Flow on Discrete Space Curves

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## Abstract

The local induction equation, or the binormal flow on space curves is a well-known model of deformation of space curves as it describes the dynamics of vortex filaments, and the complex curvature is governed by the nonlinear Schrödinger equation (NLS). In this paper, we present its discrete analogue, namely, a model of deformation of discrete space curves by the discrete nonlinear Schrödinger equation (dNLS). We also present explicit formulas for both NLS and dNLS flows in terms of the  $\tau$  function of the 2-component KP hierarchy.

## 1 Introduction

The local induction equation (LIE)

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial x} \times \frac{\partial^2 \gamma}{\partial x^2}, \quad (1)$$

is one of the most important models of deformation of space curves, where  $\gamma(x, t) \in \mathbb{R}^3$  is a smooth space curve parametrized by the arc-length  $x$  and  $t$  is a deformation parameter [6, 13, 16]. In a physical setting, it describes the dynamics of vortex filaments driven by the self-induction in the inviscid fluid under the local induction approximation [6].

It is well-known that if  $\gamma$  obeys LIE, then the curvature and the torsion, or equivalently, the complex curvature of  $\gamma$  solves the *nonlinear Schrödinger equation* (NLS) which is one of the most typical equations in the integrable systems. To show this, we use the Frenet frame  $\Phi = \Phi(x, t) =$

$[T(x, t), N(x, t), B(x, t)] \in \text{SO}(3)$ , where  $T, N, B$  are the tangent, the normal, and the binormal vectors defined by

$$T = \gamma', \quad N = \frac{\gamma''}{|\gamma''|}, \quad B = T \times N, \quad ' = \frac{\partial}{\partial x}, \quad (2)$$

respectively. Note that it follows that  $|T| = |\gamma'| = 1$  since  $x$  is the arc-length. Then we have the *Frenet-Serret formula*

$$\frac{\partial \Phi}{\partial x} = \Phi L, \quad L = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix}, \quad (3)$$

where  $\kappa = |\gamma''|$  and  $\lambda = -\langle B', N \rangle$  are the curvature and the torsion, respectively. In this setting, LIE (1) is expressed as the deformation by the binormal flow

$$\frac{\partial \gamma}{\partial t} = \kappa B, \quad (4)$$

and the corresponding deformation equation of the Frenet frame is given by

$$\frac{\partial \Phi}{\partial t} = \Phi M, \quad M = \begin{bmatrix} 0 & \kappa \lambda & -\kappa' \\ -\kappa \lambda & 0 & -\frac{\kappa''}{\kappa} + \lambda^2 \\ \kappa' & \frac{\kappa''}{\kappa} - \lambda^2 & 0 \end{bmatrix}. \quad (5)$$

The compatibility condition of the system of linear partial differential equations (3) and (5)

$$\frac{\partial L}{\partial t} - \frac{\partial M}{\partial x} = LM - ML, \quad (6)$$

yields

$$\frac{\partial \kappa}{\partial t} = -2 \frac{\partial \kappa}{\partial x} \lambda - \kappa \frac{\partial \lambda}{\partial x}, \quad \frac{\partial \lambda}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\kappa''}{\kappa} + \frac{\kappa^2}{2} - \lambda^2 \right). \quad (7)$$

Introducing the *complex curvature*  $u = u(x, t) \in \mathbb{C}$  by the *Hasimoto transformation* [6]

$$u = \kappa e^{\sqrt{-1}\Lambda}, \quad \Lambda = \int^x \lambda dx, \quad (8)$$

we see that  $u$  satisfies NLS

$$\sqrt{-1} \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} |u|^2 u = 0. \quad (9)$$

Also, one can show that this deformation is isoperimetric, namely  $|\gamma'| = 1$  for all  $t$ .

Discretization of curves and their deformations preserving underlying integrable structures is an important problem in the discrete differential geometry. For example, the isoperimetric deformation of plane discrete curves described by the discrete mKdV equation (dmKdV) has been studied in [10, 12, 14]. For discrete space curves, the deformations by the discrete sine-Gordon equation (dsG) and dmKdV has been studied in [4, 11, 12], and the deformation by dNLS is formulated in [9, 17].

The dsG and dmKdV describe torsion-preserving isoperimetric and equidistant deformation of the space discrete curves with constant torsion. However, formulation of discrete deformation of space discrete curves with varying torsion is a difficult problem. The only example so far is

presented by Hoffmann [8, 9], where he has claimed that composition of certain two isoperimetric equidistant deformations can be regarded as a discrete analogue of LIE. Also, it was used for numerical simulation of fluid flow [17, 19]. This formulation uses quaternions and its geometric meaning is clear, but description of the deformation parameters in terms of the complex curvature, thus the relation to dNLS are rather indirect.

In this paper, we present a formulation of the dNLS flow on discrete space curves from different approach; the deformation of curves is expressed in terms of the discrete Frenet frame with the coefficients given by the curvature and torsion of the curves explicitly. In this approach, dNLS arises as the equation governing the complex curvature of curves, which is the same as the case of smooth curves. Based on this formulation, we present explicit formulas for the NLS flow for smooth curves and the dNLS flow to discrete curves in terms of  $\tau$  functions of the two-component KP hierarchy by applying the theory of integrable systems. We expect that our dNLS flow can be an alternative to Hoffmann's formulation when it is used to simulate the dynamics of fluids. Also, explicit expression of the scheme and exact solutions may promote further development of theoretical studies of discrete dynamics of discrete curves from both mathematical and physical point of view.

## 2 dNLS flow on discrete space curves

Let  $\gamma_n \in \mathbb{R}^3$  be a discrete space curve with

$$|\gamma_{n+1} - \gamma_n| = \epsilon, \quad (10)$$

where  $\epsilon$  is a constant. We introduce the *discrete Frenet frame*  $\Phi_n = [T_n, N_n, B_n] \in \text{SO}(3)$  by

$$T_n = \frac{\gamma_{n+1} - \gamma_n}{\epsilon}, \quad B_n = \frac{T_{n-1} \times T_n}{|T_{n-1} \times T_n|}, \quad N_n = B_n \times T_n. \quad (11)$$

Then it follows that the discrete Frenet frame satisfies the *discrete Frenet-Serret formula*

$$\Phi_{n+1} = \Phi_n L_n, \quad L_n = R_1(-\nu_{n+1}) R_3(\kappa_{n+1}), \quad (12)$$

where

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}, \quad R_3(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (13)$$

and  $\nu_n, \kappa_n$  are defined by

$$\begin{aligned} \langle T_{n-1}, T_n \rangle &= \cos \kappa_n, & \langle B_n, B_{n-1} \rangle &= \cos \nu_n, & \langle B_n, N_{n-1} \rangle &= \sin \nu_n, \\ -\pi &\leq \nu_n < \pi, & 0 &< \kappa_n < \pi. \end{aligned} \quad (14)$$

In order to formulate a “good” discrete deformation (discretization of time), we resort to the theory of discrete integrable systems to preserve integrable nature of the NLS flow (4). As a discrete analogue of NLS (9), we consider

$$\begin{aligned} \left( \sqrt{-1} \frac{\epsilon^2}{\delta} - 1 \right) u_n^{m+1} - \left( \sqrt{-1} \frac{\epsilon^2}{\delta} + 1 \right) u_n^m + (u_{n+1}^m + u_{n-1}^{m+1})(1 + \epsilon^2 |u_n^m|^2) \Gamma_n^m &= 0, \\ \frac{\Gamma_{n+1}^m}{\Gamma_n^m} &= \frac{1 + \epsilon^2 |u_n^m|^2}{1 + \epsilon^2 |u_n^{m+1}|^2}, \end{aligned} \quad (15)$$

which we refer to as the *discrete nonlinear Schrödinger equation* (dNLS) [1, 2, 18]. Here,  $u_n^m \in \mathbb{C}$ ,  $\Gamma_n^m \in \mathbb{R}$ ,  $n$  is the space discrete variable which corresponds to the label of vertices of discrete curves,  $m$  is the discrete variable corresponding to the step of deformation,  $\epsilon$  and  $\delta$  are constants which are the lattice intervals of  $n$  and  $m$ , respectively. Moreover,  $u_n^m$  is the complex discrete curvature defined by

$$u_n^m = \frac{1}{\epsilon} \tan \frac{\kappa_n^m}{2} e^{\sqrt{-1}\Lambda_n^m}, \quad \Lambda_n^m - \Lambda_{n-1}^m = -\nu_n^m, \quad (16)$$

We impose the boundary condition as

$$u_n^m \rightarrow 0 \ (n \rightarrow \pm\infty), \quad \Gamma_n^m \rightarrow \Gamma_{\pm\infty} \ (\text{const.}) \ (n \rightarrow \pm\infty). \quad (17)$$

Then one of the main statements of this paper is given as follows:

**Theorem 2.1** (dNLS flow).

For a fixed  $m$ , let  $\gamma_n^m \in \mathbb{R}^3$  be a discrete space curve satisfying

$$|\gamma_{n+1}^m - \gamma_n^m| = \epsilon, \quad (18)$$

and  $\Phi_n^m = [T_n^m, N_n^m, B_n^m] \in \text{SO}(3)$  be the discrete Frenet frame defined in (11) satisfying the discrete Frenet-Serret formula

$$\Phi_{n+1}^m = \Phi_n^m L_n^m, \quad L_n^m = R_1(-\nu_{n+1}^m) R_3(\kappa_{n+1}^m). \quad (19)$$

Let  $u_n^m$  be a complex discrete curvature of  $\gamma_n^m$ . We determine  $u_n^{m+1}$  by dNLS (15) under the boundary condition (17) and put  $u_n^{m+1} = \frac{1}{\epsilon} \tan \frac{\kappa_n^{m+1}}{2} e^{\sqrt{-1}\Lambda_n^{m+1}}$ . We define a new curve  $\gamma_n^{m+1} \in \mathbb{R}^3$  by

$$\frac{\gamma_n^{m+1} - \gamma_n^m}{\delta} = \frac{2}{\epsilon^3} (P_n^m T_n^m + Q_n^m N_n^m + R_n^m B_n^m), \quad (20)$$

$$\begin{aligned} P_n^m &= \delta \left( -1 + \frac{\Gamma_n^m}{\cos^2 \frac{\kappa_n^m}{2}} \right), \\ Q_n^m &= \delta \left[ \tan \frac{\kappa_n^m}{2} - \tan \frac{\kappa_{n-1}^{m+1}}{2} \cos(\Lambda_{n-1}^{m+1} - \Lambda_n^m) \frac{\Gamma_n^m}{\cos^2 \frac{\kappa_n^m}{2}} \right], \\ R_n^m &= \epsilon^2 \tan \frac{\kappa_n^m}{2} - \delta \tan \frac{\kappa_{n-1}^{m+1}}{2} \sin(\Lambda_{n-1}^{m+1} - \Lambda_n^m) \frac{\Gamma_n^m}{\cos^2 \frac{\kappa_n^m}{2}}. \end{aligned} \quad (21)$$

Suppose that  $\Gamma_\infty$  and  $\Gamma_{-\infty}$  are either 1 or  $1 + \frac{\epsilon^4}{\delta^2}$ . Then, it follows that

- (1)  $|\gamma_{n+1}^{m+1} - \gamma_n^{m+1}| = \epsilon$ . Namely,  $\gamma_n^{m+1}$  is an isoperimetric deformation of  $\gamma_n^m$ .
- (2)  $u_n^{m+1}$  gives the complex discrete curvature of  $\gamma_n^{m+1}$ .

**Remark 2.2.**

- (1) The deformation (21) is not an equidistant deformation in contrast with the deformation described by dmKdV [11]. In fact, one can show that

$$\left| \frac{\gamma_n^{m+1} - \gamma_n^m}{\delta} \right|^2 = \frac{4}{\epsilon^2} \left( -1 + \frac{\Gamma_n^m}{\cos^2 \frac{\kappa_n^m}{2}} \right). \quad (22)$$

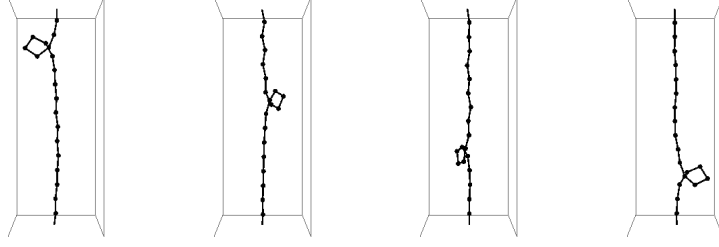


Figure 1: Numerical simulation of dNLS flow.

Equation (22) also implies that the solution of dNLS (15) should satisfy the condition  $\Gamma_n^m \geq \cos^2 \frac{\kappa_n^m}{2}$  in order to be consistent with the curve deformation. Note that this property does not contradict with Hoffmann's formulation where the deformation is constructed as composition of two isoperimetric and equidistant deformations.

- (2) Continuous limit with respect to time can be simply taken as  $t = m\delta$  and  $\delta \rightarrow 0$ . Then dNLS (15) and corresponding deformation equation (20) and (21) yields the semi-discrete NLS equation or the Ablowitz-Ladik equation [1, 2]

$$\sqrt{-1} \frac{du_n}{dt} + \frac{u_{n+1} - 2u_n + u_{n-1}}{\epsilon^2} + (u_{n+1} + u_{n-1})|u_n|^2 = 0, \quad (23)$$

and the deformation equation of discrete space curves [3, 7, 15]

$$\frac{d}{dt} \gamma_n = \frac{2}{\epsilon} \tan \frac{\kappa_n}{2} B_n. \quad (24)$$

The dNLS flow (20) and (21) implies the deformation of Frenet frame as

$$\begin{aligned} \Phi_n^{m+1} &= \Phi_n^m M_n^m, \\ M_n^m &= \frac{1}{\Gamma_{n+1}^m} \begin{bmatrix} |\alpha_n^m|^2 - |\beta_n^m|^2 & 2\Re(\alpha_n^m \beta_n^{m*}) & -2\Im(\alpha_n^m \beta_n^{m*}) \\ -2\Re(\alpha_n^m \beta_n^m) & \Re(\alpha_n^{m2} - \beta_n^{m2}) & -\Im(\alpha_n^{m2} + \beta_n^{m2}) \\ -2\Im(\alpha_n^m \beta_n^m) & \Im(\alpha_n^{m2} - \beta_n^{m2}) & \Re(\alpha_n^{m2} + \beta_n^{m2}) \end{bmatrix} \in \text{SO}(3), \end{aligned} \quad (25)$$

where  $\alpha_n^m, \beta_n^m \in \mathbb{C}$  are given by

$$\begin{aligned} \alpha_n^m &= \sqrt{-1} \frac{\delta}{\epsilon^2} \left[ \left( 1 - \sqrt{-1} \frac{\epsilon^2}{\delta} \right) - (1 + \epsilon^2 u_{n+1}^m u_n^{m+1*}) \Gamma_{n+1}^m \right] e^{\frac{\sqrt{-1}}{2} (\Lambda_n^{m+1} - \Lambda_n^m)}, \\ \beta_n^m &= \sqrt{-1} \frac{\delta}{\epsilon} (u_n^{m+1} - u_{n+1}^m) \Gamma_{n+1}^m e^{-\frac{\sqrt{-1}}{2} (\Lambda_n^m + \Lambda_n^{m+1})}, \end{aligned} \quad (26)$$

respectively. Here,  $*$  means the complex conjugate. The Frenet-Serret formula (19) and the deformation equation (25) can be transformed to the SU(2) version by the standard correspondence of SO(3) and SU(2) as

$$\begin{aligned} \phi_{n+1}^m &= \phi_n^m L_n^m, \quad L_n^m = \begin{bmatrix} \cos \frac{\kappa_{n+1}^m}{2} e^{-\frac{\sqrt{-1}}{2} \nu_{n+1}^m} & -\sin \frac{\kappa_{n+1}^m}{2} e^{-\frac{\sqrt{-1}}{2} \nu_{n+1}^m} \\ \sin \frac{\kappa_{n+1}^m}{2} e^{\frac{\sqrt{-1}}{2} \nu_{n+1}^m} & \cos \frac{\kappa_{n+1}^m}{2} e^{\frac{\sqrt{-1}}{2} \nu_{n+1}^m} \end{bmatrix}, \\ \phi_n^{m+1} &= \phi_n^m M_n^m, \quad M_n^m = \frac{1}{\sqrt{\Gamma_{n+1}^m}} \begin{bmatrix} \alpha_n^m & \beta_n^m \\ -\beta_n^{m*} & \alpha_n^{m*} \end{bmatrix}, \end{aligned} \quad (27)$$

which is known as the *Lax pair* of dNLS [1, 2]. In fact, one can verify that the compatibility condition  $L_n^m M_{n+1}^m = M_n^m L_{n+1}^{m+1}$  yields dNLS (15).

**Outline of the proof of Theorem 2.1** The first statement may be verified directly in principle, by computing  $\gamma_{n+1}^{m+1} - \gamma_n^{m+1}$  and its length from (20), (21) and the discrete Frenet-Serret formula (19) under the assumption that  $u_n^{m+1}$  is determined by dNLS (15). However, this computation is hopelessly complicated to carry out. To make it feasible, we change the Frenet frame to a different frame used in [6, 13], which we call the *complex parallel frame* in this paper. Let  $F_n^m = [T_n^m, U_n^m, U_n^{m*}] \in \text{U}(3)$  be the complex parallel frame defined by

$$U_n^m = \frac{e^{\sqrt{-1}\Lambda_n^m}}{\sqrt{2}} (N_n^m + \sqrt{-1}B_n^m). \quad (28)$$

Note that it is related to the discrete Frenet frame  $\Phi_n^m$  as

$$F_n^m = \Phi_n^m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \sqrt{-1} & -\sqrt{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{e^{\sqrt{-1}\Lambda_n^m}}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{e^{-\sqrt{-1}\Lambda_n^m}}{\sqrt{2}} \end{bmatrix}. \quad (29)$$

Then the complex curvature  $u_n^m$  naturally arises in this framework; the discrete Frenet-Serret formula (19) and the deformation of the discrete curve are rewritten in terms of  $u_n^m$  as

$$F_{n+1}^m = F_n^m X_n^m, \quad X_n^m = \frac{1}{1 + \epsilon^2 |u_{n+1}^m|^2} \begin{bmatrix} 1 - \epsilon^2 |u_{n+1}^m|^2 & -\sqrt{2}\epsilon u_{n+1}^m & -\sqrt{2}\epsilon u_{n+1}^{m*} \\ \sqrt{2}\epsilon u_{n+1}^{m*} & 1 & -\epsilon^2 (u_{n+1}^{m*})^2 \\ \sqrt{2}\epsilon u_{n+1}^m & -\epsilon^2 (u_{n+1}^m)^2 & 1 \end{bmatrix}, \quad (30)$$

and

$$\gamma_n^{m+1} = \gamma_n^m + \frac{2\delta^2}{\epsilon^3} F_n^m \begin{bmatrix} -1 + (1 + \epsilon^2 |u_n^m|^2) \Gamma_n^m \\ \frac{\epsilon}{\sqrt{2}} \left\{ (1 - \sqrt{-1} \frac{\epsilon^2}{\delta}) u_n^{m*} - u_{n-1}^{m+1*} (1 + \epsilon^2 |u_n^m|^2) \Gamma_n^m \right\} \\ \frac{\epsilon}{\sqrt{2}} \left\{ (1 + \sqrt{-1} \frac{\epsilon^2}{\delta}) u_n^m - u_{n-1}^{m+1} (1 + \epsilon^2 |u_n^m|^2) \Gamma_n^m \right\} \end{bmatrix}, \quad (31)$$

respectively. The following lemma plays a crucial role in the proof:

**Lemma 2.3.** *Let  $\gamma_n^m \in \mathbb{R}^3$  be the family of discrete space curves given in Theorem 2.1. Then it follows that*

$$|\alpha_n^m|^2 + |\beta_n^m|^2 = \Gamma_{n+1}^m. \quad (32)$$

By using (32), we have after long but straightforward calculations

$$T_n^{m+1} = \frac{\gamma_{n+1}^{m+1} - \gamma_n^{m+1}}{\epsilon} = F_n^m \frac{1}{\Gamma_{n+1}^m} \begin{bmatrix} |\alpha_n^m|^2 - |\beta_n^m|^2 \\ -\sqrt{2}\alpha_n^{m*} \beta_n^{m*} e^{-\sqrt{-1}\Lambda_n^m} \\ -\sqrt{2}\alpha_n^m \beta_n^m e^{\sqrt{-1}\Lambda_n^m} \end{bmatrix}, \quad (33)$$

from which we obtain

$$\left| \frac{\gamma_{n+1}^{m+1} - \gamma_n^{m+1}}{\epsilon} \right|^2 = \frac{\left( |\alpha_n^m|^2 - |\beta_n^m|^2 \right)^2 + 4 |\alpha_n^m|^2 |\beta_n^m|^2}{\Gamma_{n+1}^m{}^2} = \left( \frac{|\alpha_n^m|^2 + |\beta_n^m|^2}{\Gamma_{n+1}^m} \right)^2 = 1.$$

This proves the first statement. The second statement is proved as follows. Starting from  $T_n^{m+1}$  (33), we have  $B_n^{m+1}$  and  $N_n^{m+1}$  in terms of  $F_n^m$  by using (11). Then we obtain an expression of  $\Phi_n^{m+1} = [T_n^{m+1}, N_n^{m+1}, B_n^{m+1}]$  in terms of  $F_n^m$ , which can be rewritten as  $F_n^{m+1} = F_n^m Y_n^m$  with a certain matrix  $Y_n^m \in \text{U}(3)$  by using (29). This can be also transformed to the deformation equation of discrete Frenet frame of the form  $\Phi_n^{m+1} = \Phi_n^m M_n^m$  with  $M_n^m$  given in (25). Finally one can check that  $\Phi_n^{m+1}$  satisfies the discrete Frenet-Serret formula (19) for  $\kappa_n^{m+1}$  and  $\nu_n^{m+1}$  determined from the complex curvature  $u_n^{m+1}$ . This completes the proof of Theorem 2.1.  $\square$

### 3 Explicit formulas

The formulation of NLS and dNLS flows in terms of the Frenet frame enables us to apply the theory of integrable systems. As an example, we here present explicit formulas of the NLS and dNLS flows in terms of the  $\tau$  functions. For the case of plane curves, see [10]. These formulas are established based on the bilinear formalism in the theory of integrable systems by applying suitable reductions and imposing complex structure to  $\tau$  functions of the 2-component KP hierarchy, but here we only show the results, leaving full derivations to the forthcoming publications.

For any  $N \in \mathbb{N}$ , we first introduce the following three determinants, a  $2N \times 2N$  determinant  $\tau$ , two  $(2N + 1) \times (2N + 1)$  determinants  $\sigma$  and  $\rho$  as

$$\tau = \begin{vmatrix} m_{11}^{(1)} & \cdots & m_{1N}^{(1)} & 1 & \cdots & \emptyset \\ \vdots & \cdots & \vdots & & \ddots & \\ m_{N1}^{(1)} & \cdots & m_{NN}^{(1)} & \emptyset & \cdots & 1 \\ -1 & & \emptyset & m_{11}^{(2)} & \cdots & m_{1N}^{(2)} \\ & \ddots & & \vdots & \cdots & \vdots \\ \emptyset & & -1 & m_{N1}^{(2)} & \cdots & m_{NN}^{(2)} \end{vmatrix}, \quad (34)$$

$$\sigma = \begin{vmatrix} m_{11}^{(1)} & \cdots & m_{1N}^{(1)} & 1 & \cdots & \emptyset & \varphi_1^{(1)} \\ \vdots & \cdots & \vdots & & \ddots & & \vdots \\ m_{N1}^{(1)} & \cdots & m_{NN}^{(1)} & \emptyset & \cdots & 1 & \varphi_N^{(1)} \\ -1 & & \emptyset & m_{11}^{(2)} & \cdots & m_{1N}^{(2)} & 0 \\ & \ddots & & \vdots & \cdots & \vdots & \vdots \\ \emptyset & & -1 & m_{N1}^{(2)} & \cdots & m_{NN}^{(2)} & 0 \\ 0 & \cdots & 0 & \varphi_1^{(2)} & \cdots & \varphi_N^{(2)} & 0 \end{vmatrix}, \quad (35)$$

$$\rho = \left[ \begin{array}{ccc|ccc|c} m_{11}^{(1)} & \cdots & m_{1N}^{(1)} & 1 & & \emptyset & 0 \\ \vdots & & \vdots & & \ddots & & \vdots \\ m_{N1}^{(1)} & \cdots & m_{NN}^{(1)} & \emptyset & & 1 & 0 \\ \hline -1 & & \emptyset & m_{11}^{(2)} & \cdots & m_{1N}^{(2)} & \psi_1^{(1)} \\ & & \ddots & \vdots & & \vdots & \vdots \\ \emptyset & & -1 & m_{N1}^{(2)} & \cdots & m_{NN}^{(2)} & \psi_N^{(1)} \\ \hline \psi_1^{(2)} & \cdots & \psi_N^{(2)} & 0 & \cdots & 0 & 0 \end{array} \right], \quad (36)$$

where  $\emptyset$  is the empty block. Then the formulas for NLS flow on smooth curves and dNLS flow on discrete curves are obtained by choosing the entries of determinant as follows:

**NLS flow on smooth curves** We choose the entries of determinants as

$$\begin{aligned} \psi_i^{(1)} &= 1, \quad \psi_i^{(2)} = -\left(-\frac{1}{p_i^*}\right)^n e^{\xi_i^*}, \\ \varphi_i^{(1)} &= p_i^n e^{\xi_i}, \quad \varphi_i^{(2)} = -1, \\ m_{ij}^{(1)} &= -\frac{\varphi_i^{(1)} \psi_j^{(2)}}{p_i + p_j}, \quad m_{ij}^{(2)} = \frac{1}{p_i^* + p_j}, \\ \xi_i &= p_i x - \sqrt{-1} p_i^2 t + \frac{1}{p_i} z + \xi_i^{(0)}, \quad p_i, \xi_i^{(0)} \in \mathbb{C}, \end{aligned} \quad (37)$$

so that we write  $\tau = \tau_n(x, t; z)$ ,  $\sigma = \sigma_n(x, t; z)$ ,  $\rho = \rho_n(x, t; z)$ . Here,  $n$  and  $z$  are regarded as auxiliary variables. Putting

$$F = \tau_0, \quad G = -\rho_0, \quad h = -\sigma_{-2}, \quad (38)$$

we have:

**Theorem 3.1** (Explicit formula for NLS flow).<sup>1</sup>

(1) Let  $u = u(x, t) \in \mathbb{C}$  be

$$u = 2 \frac{G}{F}. \quad (39)$$

Then  $u$  satisfies NLS (9).

(2) Let  $\gamma = \gamma(x, t) \in \mathbb{R}^3$  be

$$\gamma = \left[ \begin{array}{c} \frac{h + h^*}{F} \\ \frac{1}{\sqrt{-1}} \frac{h - h^*}{F} \\ 2 \frac{\partial}{\partial z} (\log F) - x \end{array} \right]. \quad (40)$$

Then  $\gamma$  satisfies the Frenet-Serret formula (3) and the deformation equation (4).

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<sup>1</sup>The  $N$ -soliton solution for the tangent vector has been constructed by using the bilinear formalism in [5].



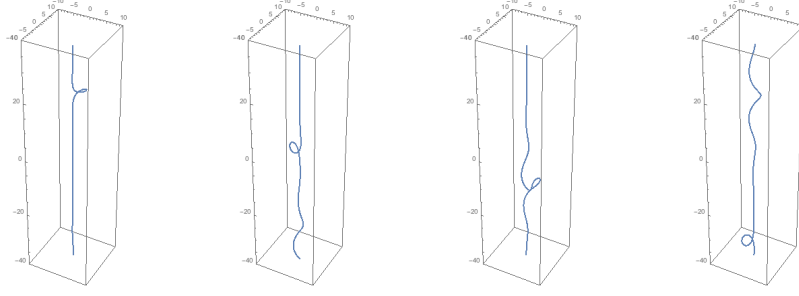


Figure 2: Interaction of loops of smooth curve by NLS flow obtained from Theorem 3.1.

**dNLS flow on discrete curves** We choose the entries of determinants as

$$\begin{aligned}
 \varphi_i^{(1)} &= p_i^{-n} e^{\zeta_i} (1 - ap_i)^{-m} (1 - cp_i)^{-r}, \quad \varphi_i^{(2)} = -\left(1 - \frac{a}{p_i}\right)^m \left(1 - \frac{1}{cp_i}\right)^s, \\
 \psi_i^{(1)} &= (1 - ap_i^*)^{-m} \left(1 - \frac{p_i^*}{c}\right)^{-s}, \quad \psi_i^{(2)} = -(p_i^*)^{-n} e^{\zeta_i^*} \left(1 - \frac{a}{p_i^*}\right)^m \left(1 - \frac{c}{p_i^*}\right)^r, \\
 m_{ij}^{(1)} &= -\frac{\varphi_i^{(1)} \psi_j^{(2)}}{p_i - \frac{1}{p_j^*}}, \quad m_{ij}^{(2)} = -\frac{\psi_i^{(1)} \varphi_j^{(2)}}{p_i^* - \frac{1}{p_j}}, \quad e^{\zeta_i} = e^{\frac{1}{2} \frac{1+cp_i}{1-cp_i} z}, \\
 a &\in \mathbb{R}, \quad p_i, c \in \mathbb{C}, \quad |c| = 1,
 \end{aligned} \tag{41}$$

so that we write  $\tau = \tau_n^m(r, s; z)$ ,  $\sigma = \sigma_n^m(r, s; z)$ ,  $\rho = \rho_n^m(r, s; z)$  with  $r, s$  and  $z$  being auxiliary variables. Putting

$$F_n^m = \tau_n^m(0, 0; z), \quad G_n^m = \rho_n^m(0, 0; z), \quad h_n^m = c^{-n} \sigma_n^m(1, -1; z), \tag{42}$$

$$a = \left(1 + \frac{\epsilon^4}{\delta^2}\right)^{\frac{1}{2}}, \quad c = \frac{1 - \sqrt{-1} \frac{\epsilon^2}{\delta}}{a}, \tag{43}$$

we have:

**Theorem 3.2** (Explicit formula for dNLS flow).

(1) Let  $u_n^m \in \mathbb{C}$  be

$$u_n^m = \frac{(-1)^m c^{-n-2m} G_n^m}{\epsilon F_n^m}, \quad \Gamma_n^m = \frac{2a}{c + \frac{1}{c}} \frac{F_{n-1}^{m+1} F_n^m}{F_n^{m+1} F_{n-1}^m}. \tag{44}$$

Then  $u_n^m$  satisfies dNLS (15).

(2) Let  $\gamma_n^m \in \mathbb{R}^3$  be

$$\gamma_n^m = \epsilon \begin{bmatrix} (-1)^m \frac{h_n^m + h_n^{m*}}{F_n^m} \\ (-1)^m \frac{h_n^m - h_n^{m*}}{\sqrt{-1} F_n^m} \\ 2 \frac{\partial}{\partial z} (\log F_n^m) - n - 2m \end{bmatrix}. \tag{45}$$

Then  $\gamma_n^m$  satisfies the Frenet-Serret formula (19) and the deformation equation (20), (21).

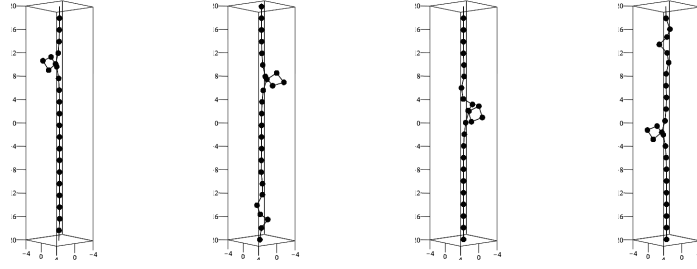


Figure 3: Interaction of loops of discrete curve by dNLS flow obtained from Theorem 3.2.

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